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Multiscale Representation of Generating and Correlation Functions for Some Models of Statistical Mechanics and Quantum Field Theory

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We consider models of statistical mechanics and quantum field theory (in the Euclidean formulation) which are treated using renormalization group methods and where the action is a small perturbation of a quadratic action. We obtain multiscale formulas for the generating and correlation functions after n renormalization group transformations which bring out the relation with the nth effective action. We derive and compare the formulas for different RGs. The formulas for correlation functions involve (1) two propagators which are determined by a sequence of approximate wave function renormalization constants and renormalization group operators associated with the decomposition into scales of the quadratic form and (2) field derivatives of the nth effective action. For the case of the block field "δ-function" RG the formulas are especially simple and for asymptotic free theories only the derivatives at zero field are needed; the formulas have been previously used directly to obtain bounds on correlation functions using information obtained from the analysis of effective actions. The simplicity can be traced to an "orthogonality-of-scales" property which follows from an implicit wavelet structure. Other commonly used RGs do not have the "orthogonality of scales" property.

KEY WORDS: Renormalization group; multiscale analysis; correlation functions; orthogonality of scales; wavelets.

1. INTRODUCTION AND RESULTS

Here we consider some statistical mechanics and quantum field theory boson and fermi models (in the Euclidean formulation) that are analyzed using renormalization group (RG) methods. We treat models whose

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actions are small perturbations of quadratic actions (see, for example, refs. 1–8). In an RG treatment a sequence of effective actions is generated by successive applications of the renormalization group transformation (RGT). We consider the cases where the RG or the RGT originates from a specified decomposition of the inverse of the operator associated with the quadratic part of the action. In the case of the block RG this decomposition is generated by the block RGT.[1,7,15] These decompositions are referred to as multiscale decompositions.

In this paper we establish identities for the generating function of the models after n RGT and by differentiating the logarithm with respect to an external field we obtain a representation for the truncated correlation functions. These representations are expressed in terms of the nth effective interaction and compared for different RGs. The point is that sufficiently good control of the effective interaction allows control of the correlation functions directly using these representations. The usual treatment of correlation functions is an analysis separate from that of the effective actions (see, for example, refs. 9–11). For the case of some IRAF lattice models, correlation functions were analyzed from this point of view in refs. 9–11. For the case of continuum quantum field theory analogous formulas can be used to obtain the short-distance behavior of correlation functions.

In our derivation we have included at each renormalization group step an additive field-independent or vacuum renormalization and also a wavefunction renormalization, thus generating a sequence of approximate wavefunction renormalization constants (denoted $\{Z_j\}$). The formulas for the c.f. have the general structure of a resummed perturbation theory. For the two-point function two propagators P_n and G_n appear (see below): P_n by itself and two G_n's as legs of a bubble. For the k-point truncated function, k G_n's appear as legs of a bubble.

We describe in more detail the propagators and bubbles. The propagators are determined by the sequence of approximate wavefunction renormalization constants and explicity known renormalization group operators associated with the decomposition into scales of the quadratic form. The bubble is determined by expectations (with respect to the nth effective action) of field derivations of the nth effective action.

Comparing the formulas for different RGs, we find that for the block spin "δ-function" RG the propagators have a remarkably simple form due to an "orthogonality-between-scales" property. This property can be traced to an implicit wavelet structure associated with this RG[13] which is not present in other commonly used RGs.

In the case of some infrared asymptotic free (IRAF) models as considered in refs. 1 and 2 and in the thermodynamic and $n \to \infty$ limits the formulas are used directly in refs. 14 and 16 to obtain bounds on correlation functions. The only information needed is obtained from the analysis of the effective actions in ref. 1. Furthermore, the long-range behavior is obtained and good bounds are obtained for the falloff of the subdominant contribution.

In this case and for the k-point c.f. we find the rather surprising result that the bubble involves only the $n \to \infty$ limit of the kth derivative of the nth effective action at zero field and there is no expectation integral. Here we have a Gaussian fixed point and quite reasonably the correlation functions are determined by the local rate of convergence of the effective action to the fixed point. Large field contributions have already been taken into account as there are large fluctuation field contributions to the sequence of approximate wavefunction renormalization constants and to the nth effective action.

For simplicity of presentation we give the results for unit lattice scalar field models. The structure of the formulas is quite general and we explain how they can be generalized to Bose and Fermi models which may be continuous.

The partition function $\varkappa$ of our models is taken to be, with $\varkappa = \varkappa(0)$,

$$\varkappa(J) = \int e^{-(\phi, \Delta\phi)/2} e^{(J,\phi)} e^{-V(\phi)} D\phi \tag{1}$$

where $D\phi = \prod_x d\phi(x)$, $V(-\phi) = V(\phi)$, and x runs over points of a finite subset of the lattice Z^d. We take $1/2(\phi, \Delta\phi)$ as our unperturbed action, where Δ is the negative of the Laplacian.

The effect of some renormalization is included in our formulas. In fact without renormalization the formulas are simple and are analogous to the formula obtained from $\varkappa(J)$ by completing the square, i.e.,

$$\varkappa(J) = e^{(J, \Delta^{-1}J)/2} \int e^{-(\phi, \Delta\phi)))/2} e^{-V(\phi + \Delta^{-1}J)} D\phi$$

and for the two-point function

$$\begin{aligned}\langle \phi(x_1)\, \phi(x_2) \rangle &= \Delta^{-1}(x_1, x_2) - \Delta^{-1}(x_1, y_1) \\ &\times \left\{ \int \left[\frac{\partial^2 V(\phi)}{\partial\phi(y_1)\, \partial\phi(y_2)} - \frac{\partial V(\phi)}{\partial\phi(y_1)} \frac{\partial V(\phi)}{\partial\phi(y_2)} \right] e^{-(\phi, \Delta\phi)/2 - V(\phi)/2} D\phi/\varkappa \right\} \\ &\times \Delta^{-1}(y_2, x_2)\end{aligned} \tag{2}$$

the difference being that V is replaced by the effective action and in the integrals Δ is replaced by the effective quadratic action. The renormalization we consider explicitly after each RGT is the separation of a constant term from the new action and most importantly a renormalization of the quadratic of the action obtained by separating out the quadratic term proportional to the unperturbed effective action. This term is included with the old quadratic part and plays the role of the unperturbed action of the next step.

The complexity of the formulas depends on the RG employed. The orthogonality between scales of the decomposition results in the simplest formulas. This orthogonality is only present in the "δ-function" block RGs which have an implicit wavelet structure.[12-14] The exponential block RG and the RG obtained from commonly used decompositions of a covariance[3,4,6,15] do not have the "orthogonality-between-scales" property. As the level of complexity of the formulas for the c.f. is the same in these two cases, we only treat the general decomposition case explicitly.

For the decomposition of the inverse of the operator associated with the unperturbed action we write

$$\Delta^{-1} = \sum_{j=0}^{n-1} \Gamma_j + \Delta_n^{-1} \tag{3}$$

We refer to this as a general decomposition. For example,

$$(\Delta^{-1})^{\wedge}(p) = \sum_{j=0}^{n-1} \int_{\alpha_j}^{\alpha_{j+1}} e^{-\alpha\mu(p)}\, d\alpha + \int_{\alpha_n}^{\infty} e^{-\alpha\mu(p)}\, d\alpha = \mu(p)^{-1} \tag{4}$$

where $\wedge$ is the Fourier transform and $\mu(p) = \sum_{\nu=1}^{d} (2 - 2\cos p_\nu)$ and $\alpha_{k+1} > \alpha_k > \alpha_0 = 0$.

For the case of the "δ-function" block RG we use the canonically-scaled block averaging operator

$$C\phi(x) = L^{(d-2)/2} L^{-d} \sum_{L/2 \leqslant y_\alpha < L/2} \phi(Lx + y) \tag{5}$$

and write $C_j = \overset{1}{C} \cdot \overset{2}{C} \cdots \overset{j}{C}$, $C_0 = I$, and $\Delta_n = (C_n \Delta^{-1} C_n^*)^{-1})^{-1}$. The RGT is defined by

$$e^{-RU(\psi)} = \int \delta(\psi - C\phi) e^{-UC(\phi)}\, D\phi, \qquad \delta(\psi - C\phi) \equiv \prod_x \delta(\psi(x) - C\phi(x)) \tag{6}$$

and the exponential block RGT is defined by

$$e^{-RU(\psi)} = \int e^{-a|\psi - C\phi|^2} e^{-U(\phi)}\, D\phi \Big/ \int e^{-a|\psi - C\phi|^2}\, D\phi \tag{7}$$

The decomposition of Δ^{-1} generated by the "δ-function" block RG is

$$\begin{aligned}\Delta^{-1} &= \sum_{j=0}^{n-1} (\Delta^{-1}C_j^*\Delta_j C_j \Delta^{-1} - \Delta^{-1}C_{j+1}^*\Delta_{j+1}C_{j+1}\Delta^{-1}) + \Delta^{-1}C_n^*\Delta_n C_n \Delta^{-1} \\ &\equiv \sum_{j=0}^{n-1} \tilde{\Gamma}_j + \tilde{\Delta}_n^{-1} \end{aligned} \tag{8}$$

which is the $a \to \infty$ singular limit of the exponential block RG decomposition. What is special about the above decomposition is that $\Delta^{1/2}\tilde{\Gamma}_j\Delta^{1/2}$ and $\Delta^{1/2}\tilde{\Delta}_n^{-1}\Delta^{1/2}$ are commuting orthogonal projections. This implies $\tilde{\Gamma}_j\Delta\tilde{\Gamma}_k = \delta_{jk}\tilde{\Gamma}_k$. Another property satisfied by the decomposition is $M_n\Delta\tilde{\Gamma}_j = 0$ for $j = 0, 1,..., n-1$, where $M_n \equiv \Delta^{-1}C_n^*\Delta_n$. These properties, related to the existence and orthogonality of wavelets, are used to simplify the formulas for the generating and correlation functions. It is shown in refs. 13 and 14 that $f_j = \Delta^{1/2}M_j u$, $Cu = 0$, is an eigenfunction of $\Delta^{1/2}\tilde{\Gamma}_j\Delta^{1/2}$, and that $h_n = \Delta^{1/2}M_n v$ is an eigenfunction of $\Delta^{1/2}\tilde{\Delta}_n^{-1}\Delta^{1/2}$ and that these functions have an interpretation in terms of lattice wavelets (see also ref. 12).

The above decomposition can also be written

$$\Delta^{-1} = \sum_{j=0}^{n-1} M_j\Gamma_j M_j^* + M_n\Delta_n^{-1}M_n^* \tag{9}$$

where $\Gamma_j \equiv \Delta_j^{-1} - \Delta_j^{-1}C^*\Delta_{j+1}C\Delta_j^{-1}$ is called the jth fluctuation covariance and where $M_k = \Delta^{-1}C_k^*\Delta_k$ is called the kth minimizer, since $\inf_{\phi: C_k\phi = \psi} \frac{1}{2}(\phi, \Delta\phi) = \frac{1}{2}(M_k\psi, \Delta M_k\psi) = \frac{1}{2}(\psi, \Delta_k\psi)$.

We now give our identities for the case of the "δ-function" and the case of a general decomposition. After n RGTs we obtain

$$\begin{aligned} z(J) = &\left\{\prod_{j=0}^{n-1} \int e^{-Z_j^{-1}(\eta_j\Delta_j\eta)/2 - V_j(M_j\eta)}\, d\mu_j(\eta)\right\} e^{(J, P_n J)/2} \\ &\times \int e^{-Z_j^{-1}(\phi, \Delta_j\phi)/2 - V_n(M_n\phi + G_n J)}\, D\phi \end{aligned} \tag{10}$$

For the "δ-function" RG, $M_k = \Delta^{-1}C_k^*\Delta_k$ and for the general decomposition, $M_k = I$. Also, $d\mu_j(\eta) = \delta(C\eta)\, D\eta$ for the "δ-function" RG and $d\mu_j(\eta) = D\eta$ for the general decomposition. $V_n(M_n\phi)$ is defined inductively by first defining $W_{n+1}(M_{n+1}\phi)$ by

$$e^{-W_{n+1}(M_{n+1}\phi)} \equiv \int e^{-Z_n^{-1}(\eta, \Delta_n\eta)/2}\, e^{-V_n(M_{n+1}\phi + M_n\eta)}\, d\mu_n(\eta)$$

and then writing

$$W_n(M_n\phi) \equiv W_n(0) + \frac{c_n}{2}(\phi, \Delta_n\phi) + V_n(M_n\phi)$$

such that we separate a field-independent term and a term proportional to the unperturbed action after n RGTs. The specific form of the operators depends on the RG and will be given below. Z_k is related to the c_j by, with $Z_0 = 1$,

$$Z_k^{-1} = 1 + c_1 + \cdots + c_k, \qquad k \geqslant 1$$

The Z_k are called approximate wavefunction renormalization constants.

P_k and G_k are propagators which can be expressed explicitly in terms of operators occurring in the decomposition of Δ^{-1}. For the "δ-function" RG

$$P_n = Z_n \Delta^{-1} - \sum_{j=0}^{n-1} \frac{(Z_n - Z_j)^2}{Z_n} \tilde{\Gamma}_j \tag{11}$$

$$G_n = Z_n \Delta^{-1} - \sum_{j=0}^{n-1} (Z_n - Z_j) \tilde{\Gamma}_j \tag{12}$$

P_n and G_n are especially simple and not mix scales. The asymptotic forms of P_n and G_n are determined by the behavior of the sequence $\{Z_j\}$. For example, if we have a bound of the type $|Z_n - Z_j| < L^{-\alpha j}$, $\alpha > 0$, typical of asymptotic free models (except ϕ_4^4) (see ref. 13) and since $|\tilde{\Gamma}_j(x, y)| < o(1) L^{-j(d-2)} \exp(-L^j |x-y|)$, then $(P_n - Z_n \Delta^{-1})(x, y)$ and $(G_n - Z_n \Delta^{-1})(x, y)$ are bounded by $c|(1 + |x-y|)^{d-2+\alpha}$, where we have used the bound

$$\sum_{j=0}^{\infty} L^{-\gamma j} \exp - (L^{-j} |x-y|) \leqslant c \frac{1}{(1 + |x-y|)^{\gamma}}$$

Thus the sums have faster decay than Δ^{-1}.

For the case of a general decomposition

$$P_n = \sum_{j=0}^{n-1} (Z_k A_k^* \Gamma_k A_k - c_{k+1} D_{k+1}^* \Delta_{k+1} D_{k+1}) + Z_n A_n^* \Delta_n^{-1} A_n \tag{13}$$

$$G_n = D_n + Z_n \Delta_n^{-1} A_n \tag{14}$$

where $A_0 = I$, $D_0 = 0$, and

$$A_m = (I \quad 0) \prod_{j=0}^{m-1} (I + R_j) \begin{pmatrix} I \\ 0 \end{pmatrix}$$

$$D_m = (0 \quad I) \prod_{j=0}^{m-1} (I + R_j) \begin{pmatrix} I \\ 0 \end{pmatrix}$$

R is the 2×2 block operator given by

$$R_n = \begin{pmatrix} -c_{n+1} \Delta_{n+1} Z_n \Gamma_n & \vdots & -c_{n+1} \Delta_{n+1} \\ \cdots & \vdots & \cdots \\ Z_n \Gamma_n & \vdots & 0 \end{pmatrix}$$

To understand the behavior of P_n and G_n, approximate A_m and D_m by the linear terms in R_k and approximate R_j by

$$R_j = \begin{pmatrix} 0 & \vdots & 0 \\ \cdots & \vdots & \cdots \\ Z_j \Gamma_j & \vdots & 0 \end{pmatrix}$$

to get

$$\begin{pmatrix} A_m \\ D_m \end{pmatrix} \approx \left(I + \sum_{j=0}^{m-1} R_j \right) \begin{pmatrix} I \\ 0 \end{pmatrix} \approx \left(\sum_{j=0}^{m-1} Z_j \Gamma_j \right)$$

which upon substituting in Eqs. (13) and (14) gives

$$P_n \approx \sum_{j=0}^{n-1} Z_j \Gamma_j + Z_n \Delta_n^{-1}$$

and

$$G_n \approx \sum_{j=0}^{n-1} Z_j \Gamma_j + Z_n \Delta_n^{-1}$$

Comparing with the decomposition of Δ^{-1} of Eq. (3), assuming the subdominance of the remainders and convergence of $\{Z_n\}$ to Z, then P_n and G_n behave like $Z\Delta^{-1}$ asymptotically.

In the case of an orthogonal decomposition, for example, $\Gamma_n^{\wedge} \Gamma_n^{\wedge} = \delta_{nm} \Gamma_m^{\wedge 2}$ and $\Gamma_m^{\wedge} \Delta_n^{\wedge} = 0$, $m < n$, then only the off-diagonal terms occur in R_m and only linear terms in R_j occur in A_m and D_m, since $R_k R_l \binom{I}{0} = 0$, and the P_n and G_n simplify to the G_n of the "δ-function" RG. One way to realize an orthogonal decomposition of the convariance is to use disjoint characteristic functions in the Fourier transform of Δ^{-1}, but this is difficult to control due to the lack of exponential or good polynomial decay in position space.

We now turn to a derivation of a representation for correlation functions. From (10), after expanding $V_n(M_n \phi_n + G_n J)$ as

$$V_n(M_n \phi + G_n J) = \sum_{l=0}^{\infty} \frac{1}{l!} D^l_{y_1 \cdots y_l} V_n(M_n \phi) \prod_{i=1}^{l} G_n(y_i, x_i)\, J(x_i)$$

where we have set

$$D^l_{y_1\cdots y_l} V_n(M_n\phi) \equiv \partial^l V_n(\chi = M_n\phi)/\partial\chi(y_1)\cdots\partial\chi(y_l)$$

we can write the k-point truncated correlation function as

$$\begin{aligned}\left\langle \prod_{i=1}^{k} \phi(x_i)\right\rangle^T &\\ &\equiv \partial^k \ln z(J)/\partial J(x_1)\cdots\partial J(x_k)|_{J=0}\\ &= \delta_{k2} P_n(x_1, x_2) - \sum_{\pi=\{I_i\}_{i=1}^{\tau}} (-1)^{-1} \left\langle \prod_{i=1}^{\tau} D^{|I_i|}_{y_{I_i}} V_n(M_n\phi)\, G_n^{|I_i|}(Y_{I_i}. X_{I_i})\right\rangle^T_{V_n}\end{aligned} \tag{15}$$

where $\pi = \{I_i\}$ is the set of partitions of $\{1, 2,..., k\}$ and $\langle\cdot\rangle^T_{V_n}$ is the truncated expectation with probability measure

$$N^{-1} \exp[-V_n(M_n\phi) - \tfrac{1}{2} Z_n^{-1}(\phi, \Delta_n\phi)]\, D\phi$$

and we have set

$$G_n^{|I_i|}(y_{I_i}, x_{I_i}) = \prod_{j=1}^{|I_i|} G_n(u_j, v_j)$$

for $u_j \subset x_{I_i}$ and $v_i \subset y_{I_i}$. For example, the two-point function is

$$\begin{aligned}\langle \phi(x_1)\,\phi(x_2)\rangle = P_n(x_1, x_2) - G_n(x_1, y_1)\langle D^2_{y_1 y_2} V_n(M_n\phi\ )&\\ - D_{y_1} V_n(M_n\phi)\, D_{y_2} V_n(M_n\phi)\rangle^T_{V_n} G_n(y_2, x_2)&\end{aligned} \tag{16}$$

which can be compared with Eq. (2).

If we are considering IRAF models it is more convenient to separate out the part of $V_n(M_n\phi + G_n J)$ which is ϕ independent, writing

$$\begin{aligned}V_n(M_n\phi + G_n J) = \sum_{l=0}^{\infty} \frac{1}{l!} D^l_{y_1\cdots y_l} V_n(0) \prod_{i=1}^{l} G_n(y_i, x_i)\, J(x_i)&\\ + \sum_{l=0}^{\infty} \frac{1}{l!} [D^l_{y_1\cdots y_l} V_n(M_n(\phi)) - D^l_{y_1\cdots y_l} V_n(0)]&\\ \times \prod_{i=1}^{l} G_n(y_i, x_i)\, J(x_i)&\end{aligned}$$

In this case

$$\left\langle \prod_{i=1}^{k} \phi(x_i) \right\rangle^T = \delta_{k2} P_n(x_1, x_2) - D^k_{y_1 \cdots y_k} V_n(0) \prod_{i=1}^{k} G_n(y_i, x_i) + R_{kn} \quad (17)$$

where R_{kn} is given by the second term in (15) with

$$D^{|I_i|}_{y_{I_i}} V_n(M_n \phi) - D^{|I_i|}_{y_{I_i}} V_n(0)$$

replacing

$$D^{|I_i|}_{y_{I_i}} V_n(M_n \phi)$$

The reason that (17) is preferred is that the numerator of the integrand of R_{kn} is field dependent and R_n is expected to vanish in the thermodynamic and $n \to \infty$ limit since it represents roughly contributions from an interval of momentum $[0, L^{-n}]$ or $[0, a_n]$ where $a_n \to 0$ as $n \to \infty$.[14,18] For the "δ-function" RGT the vanishing of R_{kn} in this limit is proved in ref. 19 using a small-large field analysis. In this limit and for $k = 2$ the limit of the first term of P_n gives the dominant long-range behavior of $Z\Delta^{-1}$, where $Z \equiv \lim_{n \to \infty} Z_n$ and the rest of P_n and the second term of (17) are shown to fall off faster than Δ^{-1}.

The proof of these results for the $\partial\phi$ model in $d \geqslant 3$ can be found in refs. 14, 16, and 19, but as the proof is so transparent, let us sketch it here. From Eq. (17) we see that we only need a bound on $Z_n - Z_j$ and

$$D^2_{y_1 y_2} V_n(0) \equiv \partial^2 V_n(x_n = M_{n\phi} = 0)/\partial \chi_n(y_1)\, \partial \chi_n(y_2)$$

These bounds are readily obtained from the analysis of the effective actions in ref. 13. From ref. 1, $|Z_n - Z_j| < c\delta^j$, $0 < \delta < 1$, so that the sum in P_n of Eq. (11) decays faster than Δ^{-1}. Also from ref. 1, $V_n(M_n \phi)$ is analytic in a neighborhood of zero and the quadratic part is given by the irrelevant term $\frac{1}{2}(\partial\partial M_n \phi,\ S_n\, \partial\partial M_n \phi)$, where we write everything on the unit lattice. Thus $D^2_{y_1 y_2} V_n(0) = (\partial^* \partial^* S_n\, \partial\partial)(y_1, y_2)$. From the iteration of the bounds of Section 6 of ref. 1 the kernel of S_n has the bound

$$|S_n(y_1, y_2)| < c/(1 + |y_1 - y_2|)^{d-2+\varepsilon}, \qquad \varepsilon > 0$$

Also $\partial\partial G_n(x, y)$ has the bound $c/(1 + |x - y|)^d$, so that the second term of Eq. (17) is bounded by

$$\begin{aligned} &|G_n\, \partial^* \partial^* S_n\, \partial\partial G_n)(x_1, x_2)| \\ &\quad \leqslant \sum_{y_1, y_2} (1 + |x_1 - y_1|)^{-d} (1 + |y_1 - y_2|)^{-(d-2+\varepsilon)} (1 + |y_2 - x_2|)^{-d} \\ &\quad \leqslant c/(1 + |x_1 - x_2|)^{d-2+\varepsilon'}, \qquad \varepsilon' > 0 \end{aligned}$$

i.e., it decays faster than Δ^{-1}. Roughly speaking, in momentum for small $|p|$ the bound is $1/|p|^{2-\varepsilon}$, which is less singular than $\tilde{\Delta}^{-1}(p) = 2/p^2$, thus leading to the faster than Δ^{-1} decay.

We describe the organization of the rest of the paper. As the identity (10) was already derived for the "δ-function" RG in refs. 14 and 15, in Section 2 we derive (10) for the general decomposition case. In Section 3 we discuss some generalizations.

2. DERIVATION OF GENERATING FUNCTION IDENTITY

Here we derive the formula (10) for the general decomposition of (3). We use the translation formula

$$\int f(\psi)\, e^{(\psi, K)}\, d\mu(\psi) = e^{(K, CK)/2} \int f(\psi + CK)\, d\mu((\psi)$$

for a Gaussian measure with covariance C. Assume after n steps that

$$\mathscr{z}(J) = \exp\left[-\sum_{i=1}^{n} W_i(0)\right] \exp[\tfrac{1}{2}(J, C_n J)]$$

$$\times \int \exp[-\tfrac{1}{2} b_n(\phi_n, \Delta_n \phi_n)] \exp[(\phi_n, A_n J)] \exp[-V_n(\phi_n + D_n J)]\, D\phi_n$$

Setting $\Delta_n^{-1} = \Delta_{n+1}^{-1} + \Gamma_n$, $\phi_n = \phi_{n+1} + \eta_n$, we obtain, after using the translation formula,

$$\mathscr{z}(J) = \exp\left[-\sum_{i=1}^{n} W_i(0)\right] \exp[\tfrac{1}{2}(J, C_n J)] \exp \frac{(J, A_n b_n^{-1} \Gamma_n A_n J)}{2}$$

$$\times \int \exp[\tfrac{1}{2} b_n(\phi_{n+1}, \Delta_{n+1} \phi_{n+1})] \exp[(\phi_{n+1}, A_n J)]$$

$$\times \exp[-W_{n+1}(\phi_{n+1} + (D_n + b_n^{-1} \Gamma_n A_n) J)]\, D\phi_{n+1}$$

where

$$\exp[-W_{n+1}(\phi_{n+1})] \equiv \int \exp[\tfrac{1}{2} b_n(\eta_n, \Gamma_n \eta_n^{-1})] \exp[-V_n(\phi_{n+1} + \eta_n)]\, D\eta_n$$

Now we renormalize. Write, defining c_{n+1} and V_{n+1} by

$$W_{n+1}(\phi_{n+1}) = W_{n+1}(0) + \frac{c_{n+1}}{2} (\phi_{n+1}, \Delta_{n+1} \phi_{n+1}) + V_{n+1}(\phi_{n+1})$$

so that

$$W_{n+1}(\phi_{n+1}+D_{n+1}J)=W_{n+1}(0)+\frac{c_{n+1}}{2}(\phi_{n+1}+D_{n+1}J,\Delta_{n+1}$$

$$\times(\phi_{n+1}+D_{n+1}J))+V_{n+1}(\phi_{n+1}+D_{n+1}J)$$

where we set $D_{n+1}=D_n+b_n^{-1}\Gamma_n A_n$. Substituting in $\mathscr{z}(J)$ gives

$$\mathscr{z}(J)=\exp\left[-\sum_{i=1}^{n+1}W_i(0)\right]$$

$$\times\exp\{\tfrac{1}{2}(J,[C_n+b_nA_n^*\Gamma_nA_n-c_{n+1}D_{n+1}^*\Delta_{n+1}D_{n+1}]J)\}$$

$$\times\int\exp[-\tfrac{1}{2}(b_n+c_{n+1})(\phi_{n+1},\Delta_{n+1}\phi_{n+1})]$$

$$\exp\{(\phi_{n+1},[A_n-c_{n+1}\Delta_{n+1}D_{n+1}]J)\}$$

$$\times\exp[-V_{n+1}(\phi_{n+1}+D_{n+1}J)]\,D\phi_{n+1}$$

$$=\exp\left[-\sum_{i=1}^{n+1}W_i(0)\right]\exp\frac{(J,C_{n+1}J)}{2}$$

$$\times\int\exp[\tfrac{1}{2}b_{n+1}(\phi_{n+1},\Delta_{n+1}\phi_{n+1})]\exp[(\phi_{n+1},A_{n+1}J)]$$

$$\times\exp[-V_{n+1}(\phi_{n+1}+D_{n+1}J)]\,D\phi_{n+1}$$

and we define D_{n+1}, C_{n+1}, b_{n+1}, and A_{n+1} by

$$\begin{aligned}
D_{n+1}&=D_n+b_n^{-1}\Gamma_nA_n, && D_0=0, && D_1=\Gamma_0\\
C_{n+1}&=C_n+b_n^{-1}A_n^*\Gamma_nA_n\\
&\quad-c_{n+1}D_{n+1}^*\Delta_{n+1}D_{n+1}, && C_0=0, && C_1=\Gamma_0-c_1\Gamma_0\Delta_1\Gamma_0\\
b_{n+1}&=b_n+c_{n+1}, && b_0=1, && b_1=1+c_1\\
A_{n+1}&=A_n-c_{n+1}\Delta_{n+1}D_{n+1}, && A_0=1, && A_1=1-c_1\Delta_1\Gamma_0
\end{aligned}$$

We solve these recursion relations later after performing the final step. Using the translation formula in the last integral above, we have

$$\mathscr{z}(J)=\exp\left[-\sum_{i=1}^{n+1}W_i(0)\right]\exp\frac{(J,[C_{n+1}+A_{n+1}^*b_{n+1}^{-1}\Delta_{n+1}A_{n+1}]J)}{2}$$

$$\times\int\exp[-\tfrac{1}{2}b_{n+1}(\phi_{n+1},\Delta_{n+1}\phi_{n+1})]$$

$$\times\exp[-V_{n+1}(\phi_{n+1}+D_{n+1}+b_{n+1}\Delta_{n+1}^{-1}A_{n+1}^{-1}J)]\,D\phi_{n+1}$$

where

$$e^{-W_{k+1}(0)}=\int e^{-b_k(\eta_k,\Gamma_k^{-1}\eta_k)/2}\, e^{-V_k(\eta_k)}\, D\eta_k$$

and we are done upon setting $P_{n+1}=C_{n+1}+A^*_{n+1}b^{-1}_{n+1}\Delta^{-1}_{n+1}A_{n+1}$ and $G_{n+1}=D_{n+1}+b^{-1}_{n+1}\Delta^{-1}_{n+1}A_{n+1}$.

Now we solve the recursion relations. For b_{n+1} and C_{n+1} we have

$$b_{n+1}=1+c_1+c_2+\cdots+c_{n+1}\equiv Z^{-1}_{n+1}$$

$$C_{n+1}=\sum_{k=0}^{n}\,(b_k^{-1}A_k^*\Gamma_kA_k-c_{k+1}D^*_{k+1}\Delta_{k+1}D_{k+1})$$

We treat the relations for A_{n+1} and D_{n+1} as a system. Write

$$A_{n+1}=A_n-c_{n+1}\Delta_{n+1}D_{n+1}=A_n-c_{n+1}\Delta_{n+1}D_n-c_{n+1}\Delta_{n+1}b_n^{-1}\Gamma_nA_n$$
$$D_{n+1}=D_n+b_n^{-1}\Gamma_nA_n$$

or, where R_n is a 2×2 block operator,

$$\begin{pmatrix}A_{n+1}-A_n\\D_{n+1}-D_n\end{pmatrix}=(R_n)\begin{pmatrix}A_n\\S_n\end{pmatrix},\qquad R_n=\left(\begin{array}{c:c}-c_{n+1}\Delta_{n+1}b_n^{-1}\Gamma_n & -c_{n+1}\Delta_{n+1}\\ \hdashline b_n^{-1}\Gamma_n & 0\end{array}\right)$$

Thus, with I a 2×2 block identity operator,

$$\begin{pmatrix}A_{n+1}\\D_{n+1}\end{pmatrix}=\prod_{j=0}^{n}\,(I+R_j)\begin{pmatrix}I\\0\end{pmatrix}$$

and we have completed the derivation of (10).

3. GENERALIZATIONS

If we consider massive lattice regularized ultraviolet problems, for example, the ϕ^4 interaction in three dimensions, in the framework of the block RG it is more convenient to start with an ε lattice and use a sequence of averaging operations as in ref. 7 which map from $L^k\varepsilon$ to $L^{k+1}\varepsilon$ lattices. RGTs are applied until a lattice of order unity is reached. Identities similar to Eqs. (10) and (15) can be obtained and the remainder represents contributions from momentum scales roughly between 0 and unity. The $\varepsilon\searrow 0$ limit gives a continuum model and the formulas can be used to analyze the short-distance behavior of correlation functions. For the

δ-function block RG continuum wavelets will then enter.[12,13] In the case of a general decomposition a continuous space covariance with ultraviolet cutoff can be used as in ref. 6.

For fermion models similar formulas to Eqs. (10) and (15) can also be obtained. For the block RG the RGT is formulated in terms of Grassman variables and decomposition formulas and properties of RG operators are obtained in ref. 15. The unperturbed action operator is taken to be the Dirac operator. The decomposition formula for the exponential block RG has the same structure as in Eq. (8), the difference being that the effective quadratic operators $\{P_n\}$ are not self-adjoint. Furthermore, in contrast to the exponential block RG, the δ-function RG does not have uniformly exponentially decaying kernels for $\{D_n\}$.[17] Thus the "orthogonality-of-scales" formulas are not obtained. Formulas for the effective potential and Schwinger functions in some fermion models have been obtained and applied in ref. 18.

For perturbations $V(\partial A)$ of the free electromagnetic field on a lattice with action $\frac{1}{2}(\partial A, \partial A)$, where ∂A is the electromagnetic field 2-form as in ref. 8, similar formulas can be obtained where in the context of the block RG the scalar field averaging operators are generalized to one- and two-form averaging operators.

We leave the application of the correlation function formulas for the case of a general decomposition to the proponents of RGs other than the "δ-function" block field RG.

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